

HIGHER DIMENSIONAL TWISTED ALEXANDER POLYNOMIALS FOR METABELIAN REPRESENTATIONS

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ABSTRACT. We study the asymptotic behavior of the twisted Alexander polynomial for the sequence of $\mathrm{SL}_n(\mathbb{C})$ -representations induced from an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of a knot group. We give the limits of the leading coefficients in the asymptotics of the twisted Alexander polynomial and related Reidemeister torsion. The concrete computations for all genus one two-bridge knots are also presented.

1. INTRODUCTION

The twisted Alexander polynomial was introduced by Lin [Lin01] and Wada [Wad94]. It is a generalization of the classical Alexander polynomial by a linear representation of a knot group. The classical Alexander polynomial contains important topological features of knots. It has been shown that the twisted Alexander polynomial gives refinements of results derived from the Alexander polynomial. On the other hand, the linear representations of a group are closely related to the geometric structures of a low-dimensional manifold. In particular, the $\mathrm{SL}_2(\mathbb{C})$ -representations of the fundamental group of a 3-manifold can be thought of as descriptions of the geometric structures, especially the hyperbolic structures, of the manifold. We are interested in finding the relationship between the twisted Alexander polynomial and the hyperbolic structures of knot exteriors.

It was shown in [MFP14, Mül12, TY] that the asymptotic behavior of the Reidemeister torsion is related to hyperbolic structures. The twisted Alexander polynomial can be regarded as a kind of Reidemeister torsion by [KL99, Kit96]. It is natural to try to relate the asymptotic behavior of the twisted Alexander polynomial to some features of hyperbolic structures as in [God].

For a metabelian representation of a knot group, its twisted Alexander polynomial has been studied in terms of the factorization of this invariant [BF14, HM, Yam13]. In this paper, using such a factorization property, we wish to investigate the limit of the twisted Alexander polynomial for a sequence of $\mathrm{SL}_n(\mathbb{C})$ -representations induced by an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of a knot group and study a relation to degeneration of hyperbolic structures. Our purpose is to provide the asymptotic behavior of *the higher dimensional twisted Alexander polynomial* for irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representations (see Definition 2.4). Then we will study a relation to degeneration of hyperbolic structures from the viewpoint of the Reidemeister torsion.

Let E_K be the knot exterior $S^3 \setminus N(K)$ of a knot K . Here $N(K)$ is an open tubular neighbourhood of K . Upon choosing a homomorphism ρ from $\pi_1(E_K)$ into $\mathrm{SL}_2(\mathbb{C})$, we have a sequence of homomorphisms $\sigma_n \circ \rho$ from $\pi_1(E_K)$ into $\mathrm{SL}_n(\mathbb{C})$ by the composition with $\sigma_n: \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SL}_n(\mathbb{C})$ which is called *the n -dimensional irreducible representation* of

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$\mathrm{SL}_2(\mathbb{C})$. We can also consider the sequence of twisted Alexander polynomials corresponding to the sequence $\rho_n = \sigma_n \circ \rho$. Our main results are stated as follows:

Theorem (Theorem 3.2). *Let ρ be an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of $\pi_1(E_K)$. Then there exists a sequence of metabelian $\mathrm{SL}_2(\mathbb{C})$ -representations ψ_1, \dots, ψ_p which gives the limit of the leading coefficient of $\log \Delta_{K, \rho_{2N}}(t)$ as*

$$\lim_{N \rightarrow \infty} \frac{\log \Delta_{K, \rho_{2N}}(t)}{2N} = \frac{1}{2p} \sum_{j=1}^p \log \Delta_{K, \psi_j}(t).$$

Remark 1.1. The number p is a divisor of $|\Delta_K(-1)|$ where $\Delta_K(t)$ is the Alexander polynomial of K .

Under the assumption that $\Delta_{K, \psi_j}(1) \neq 0$ for all $j = 1, \dots, p$ in the above theorem, the Reidemeister torsion $\mathbb{T}_{K, \rho_{2N}}$ is defined for all N . Then we relate the asymptotic behavior of the twisted Alexander polynomial to that of the Reidemeister torsion. The limit of the leading coefficient in $\mathbb{T}_{K, \rho_{2N}}$ is expressed as follows.

Corollary (Corollary 3.4). *Taking $t = 1$, we can express the limit of the leading coefficient of $\log |\mathbb{T}_{K, \rho_{2N}}|$ as*

$$\lim_{N \rightarrow \infty} \frac{\log |\mathbb{T}_{K, \rho_{2N}}|}{2N} = \frac{1}{2p} \sum_{j=1}^p \log |\Delta_{K, \psi_j}(1)|.$$

In particular, the growth order of $\log |\mathbb{T}_{K, \rho_{2N}}|$ is the same as $2N$.

For a hyperbolic knot K and its holonomy representation $\rho_{\mathrm{hol}}: \pi_1(E_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$, it was shown in [MFP14, God] that the growth order of $\log |\mathbb{T}_{K, \sigma_n \circ \rho_{\mathrm{hol}}}|$ or $\log |\Delta_{K, \sigma_n \circ \rho_{\mathrm{hol}}}(1)|$ is the same as n^2 and the leading coefficient converges to the hyperbolic volume of E_K divided by 4π . The set of $\mathrm{SL}_2(\mathbb{C})$ -representations of $\pi_1(E_K)$ can be regarded as a deformation space of hyperbolic structures of E_K . From the difference in the growth order of the Reidemeister torsion, we can say that every irreducible metabelian representation corresponds to a degenerate hyperbolic structure for any hyperbolic knot. For examples of this phenomenon, we refer to [TY].

We also give an explicit description of the asymptotic behavior of the twisted Alexander polynomial for metabelian representations of genus one two-bridge knot groups. Note that all genus one two-bridge knots are given by $J(2m, \pm 2n)$ illustrated as in Fig. 1, where m and n are positive integers.

Theorem (Theorem 4.4, 4.6 and Corollary 4.5, 4.7). *Let ρ be an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of $\pi_1(E_{J(2m, \pm 2n)})$. Then there exists a divisor q of $|\Delta_{J(2m, \pm 2n)}(-1)|$ such that*

$$\lim_{N \rightarrow \infty} \frac{\log \Delta_{J(2m, \pm 2n), \rho_{2N}}(t)}{2N} = \frac{1}{2q} \log \frac{m^2 n^2 t^4 + (2m^2 n^2 \mp 4mn + 1)t^2 + m^2 n^2}{(t^2 + 1)^2} + \frac{1}{2} \log(t^2 + 1).$$

Moreover, the leading coefficient of the logarithm of the higher dimensional Reidemeister torsion converges as

$$\lim_{N \rightarrow \infty} \frac{\log |\mathbb{T}_{J(2m, \pm 2n), \rho_{2N}}|}{2N} = \frac{1}{q} \log \frac{2mn \mp 1}{2} + \frac{1}{2} \log 2.$$

This paper is organized as follows. Section 2 provides expositions of the twisted Alexander polynomial for a linear representation and for a sequence of $\mathrm{SL}_n(\mathbb{C})$ -representations induced from an $\mathrm{SL}_2(\mathbb{C})$ -representation of a knot group. We will also touch a relation between the twisted Alexander polynomial and the Reidemeister torsion in Subsection 2.3.

In Section 3, our main results are stated and proved. Section 4 is devoted to the study of asymptotic behaviors of the twisted Alexander polynomial and the Reidemeister torsion for genus one two-bridge knots.

2. PRELIMINARIES

We will review the twisted Alexander polynomial and a Lin presentation of a knot group. Then we will also see a relation between the twisted Alexander polynomial and the Reidemeister torsion for a knot exterior.

2.1. Twisted Alexander polynomial. Throughout the paper, K denotes a knot in S^3 and E_K denotes the knot exterior $S^3 \setminus N(K)$ where $N(K)$ is an open tubular neighbourhood of K . We use the symbol α to denote the abelianization homomorphism from a knot group $\pi_1(E_K)$ onto $\langle t \rangle \simeq H_1(E_K; \mathbb{Z})$. Here t is the homology class of a meridian of the knot K . We follow the definition of the twisted Alexander polynomial in [Wad94].

We need a homomorphism ρ from the knot group $\pi_1(E_K)$ into $\mathrm{GL}_n(\mathbb{C})$, which is called a $\mathrm{GL}_n(\mathbb{C})$ -representation, to define the twisted Alexander polynomial for K . For a $\mathrm{GL}_n(\mathbb{C})$ -representation ρ of $\pi_1(E_K)$, we denote by Φ_ρ the \mathbb{Z} -linear extension of the tensor product $\alpha \otimes \rho$ defined as

$$\begin{aligned} \Phi_\rho: \mathbb{Z}[\pi_1(E_K)] &\rightarrow M_n(\mathbb{C}[t^{\pm 1}]) \\ \sum_i a_i \gamma_i &\mapsto \sum_i a_i \alpha(\gamma_i) \otimes \rho(\gamma_i). \end{aligned}$$

Here $M_n(\mathbb{C}[t^{\pm 1}])$ is the set of $n \times n$ matrices whose entries are Laurent polynomials in $\mathbb{C}[t^{\pm 1}]$ and we identify $M_n(\mathbb{C}[t^{\pm 1}])$ with the tensor product $\mathbb{C}[t^{\pm 1}] \otimes M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the set of $n \times n$ complex matrices.

Definition 2.1. Choose a presentation $\pi_1(E_K) = \langle g_1, \dots, g_k \mid r_1, \dots, r_{k-1} \rangle$ of deficiency 1 and let ρ be a $\mathrm{GL}_n(\mathbb{C})$ -representation of $\pi_1(E_K)$. Suppose that $\Phi_\rho(g_\ell - 1) \neq 0$. Then the twisted Alexander polynomial is defined as

$$\Delta_{K, \rho}(t) = \frac{\det \left(\Phi_\rho \left(\frac{\partial r_i}{\partial g_j} \right) \right)_{\substack{1 \leq i \leq k-1 \\ 1 \leq j \leq k, j \neq \ell}}}{\det(\Phi_\rho(g_\ell - 1))}$$

where $\partial r_i / \partial g_j$ is the Fox differential of r_i by g_j .

Remark 2.2. We mention the well-definedness of the twisted Alexander polynomial without proofs. For the details, see [Wad94, Theorem 1, Corollary 4].

- The twisted Alexander polynomial is independent of the choice of the presentation.
- If $\Phi_\rho(g_\ell - 1) \neq 0$ and $\Phi_\rho(g_{\ell'} - 1) \neq 0$, then it holds that

$$\frac{\det \left(\Phi_\rho \left(\frac{\partial r_i}{\partial g_j} \right) \right)_{\substack{1 \leq i \leq k-1 \\ 1 \leq j \leq k, j \neq \ell}}}{\det(\Phi_\rho(g_\ell - 1))} = \pm \frac{\det \left(\Phi_\rho \left(\frac{\partial r_i}{\partial g_j} \right) \right)_{\substack{1 \leq i \leq k-1 \\ 1 \leq j \leq k, j \neq \ell'}}}{\det(\Phi_\rho(g_{\ell'} - 1))}.$$

We will mainly consider the twisted Alexander polynomial for $\mathrm{SL}_n(\mathbb{C})$ -representations of a knot group. In particular, we deal with the situation that the $\mathrm{SL}_n(\mathbb{C})$ -representations ρ_n are induced from an $\mathrm{SL}_2(\mathbb{C})$ -representation ρ of $\pi_1(E_K)$.

There exists a sequence of homomorphisms σ_n from $\mathrm{SL}_2(\mathbb{C})$ into $\mathrm{SL}_n(\mathbb{C})$, which is referred as the n -dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$. This homomorphism

σ_n is given by the action of $\mathrm{SL}_2(\mathbb{C})$ on the vector space V_n consisting of homogeneous polynomials $p(x, y)$ with degree $n - 1$ as follows:

$$A \cdot p(x, y) = p(x', y') \quad \text{where} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, given an $\mathrm{SL}_2(\mathbb{C})$ -representation ρ of $\pi_1(E_K)$, there exists the sequence of $\mathrm{SL}_n(\mathbb{C})$ -representations $\sigma_n \circ \rho$.

Remark 2.3. If the eigenvalues of $A \in \mathrm{SL}_2(\mathbb{C})$ are $\xi^{\pm 1}$, then $\sigma_n(A)$ has the eigenvalues $\xi^{\mp(2j-1)}$ ($j = 1, \dots, n$). One can see this fact from the image by σ_n of a diagonal matrix with respect to the standard basis $\{x^{n-1}, x^{n-2}y, \dots, xy^{n-2}, y^{n-1}\}$ of V_n .

Definition 2.4. Under the assumptions of Definition 2.1, set $\rho_n = \sigma_n \circ \rho$. Suppose that $\det(\Phi_{\rho_n}(g_\ell - 1)) \neq 0$. Then the twisted Alexander polynomial for ρ_n is defined as

$$\Delta_{K, \rho_n}(t) = \frac{\det\left(\Phi_{\rho_n}\left(\frac{\partial r_i}{\partial g_j}\right)\right)_{\substack{1 \leq i \leq k-1 \\ 1 \leq j \leq k, j \neq \ell}}}{\det(\Phi_{\rho_n}(g_\ell - 1))}.$$

We call $\Delta_{K, \rho_n}(t)$ the n -dimensional twisted Alexander polynomial for ρ .

When some ℓ satisfies $\det(\Phi_{\rho_n}(g_\ell - 1)) \neq 0$ for all n , we have the sequence of the twisted Alexander polynomial $\Delta_{K, \rho_n}(t)$.

By definition, the twisted Alexander polynomial could be a rational function. There exists a sufficient condition for it to be a Laurent polynomial.

Proposition 2.5 (Proposition 8 in [Wad94]). *If a $\mathrm{GL}_n(\mathbb{C})$ -representation ρ of $\pi_1(E_K)$ satisfies that there exists an element γ in $[\pi_1(E_K), \pi_1(E_K)]$ such that $\rho(\gamma)$ does not have the eigenvalue 1, then the twisted Alexander polynomial $\Delta_{K, \rho}(t)$ is a Laurent polynomial.*

Moreover it was shown by [KM05, Theorem 3] that $\Delta_{K, \rho}(t)$ is a Laurent polynomial for all non-abelian $\mathrm{SL}_2(\mathbb{C})$ -representations ρ . Here the terminology ‘‘non-abelian’’ means that the image $\rho(\pi_1(E_K))$ is not contained in an abelian subgroup in $\mathrm{SL}_2(\mathbb{C})$. We will find rational functions in the sequence of $\Delta_{K, \rho_n}(t)$ in Section 4.

2.2. Lin presentation. In [Lin01], X.-S. Lin introduced a special presentation of a knot group by using a *free Seifert surface*. A Seifert surface is called *free* if its exterior $S^3 \setminus N(S)$ is a handlebody where $N(S)$ is an open tubular neighbourhood. This means that $\pi_1(S^3 \setminus N(S))$ is a free group if a Seifert surface S is free and the number of generators in $\pi_1(S^3 \setminus N(S))$ coincides with twice of the genus of S . It is known that every knot K has free Seifert surfaces.

Definition 2.6. Suppose that S is a free Seifert surface with genus g and $\pi_1(S^3 \setminus N(S))$ is generated by x_1, \dots, x_{2g} which are the homotopy classes corresponding to the cores of 1-handles in $S^3 \setminus N(S)$. We denote by μ a meridian of K . Then the knot group $\pi_1(E_K)$ is presented as

$$(1) \quad \pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle$$

where a_i^\pm correspond to loops given by pushing up or down the spine $\vee_{i=1}^{2g} a_i$ of S along the normal direction in $N(S)$. We call the presentation (1) the *Lin presentation* of $\pi_1(E_K)$ associated with S .

Remark 2.7. We have the linking number $\ell k(x_i, K) = 0$ for every x_i in (1). This is due to that the representative loop of x_i lies in the outside of S . This means that each x_i is a commutator in $\pi_1(E_K)$.

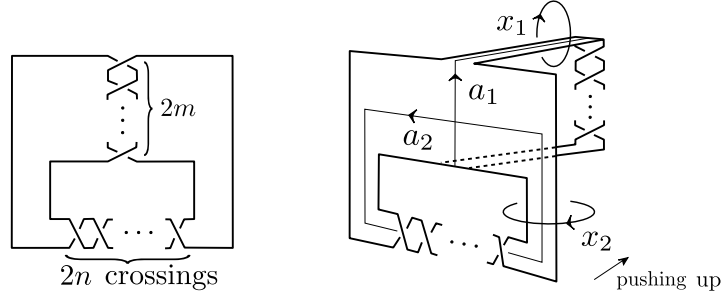


FIGURE 1. a diagram (left) and a free Seifert surface (right) of $J(2m, 2n)$

Example 2.8. The double twist knot $J(2m, 2n)$ in Hoste-Shanahan's notation [HS04] is illustrated in Figure 1. Here both m and n are positive integers. The knot $J(2m, 2n)$ corresponds to $K_{-m, -n}$ in [Lin01]. The right side in Figure 1 shows a free Seifert surface S of $J(2m, 2n)$. We can choose the loops x_1 and x_2 as generators of $\pi_1(S^3 \setminus N(S))$. The spine of S is $a_1 \vee a_2$. The loop a_1^+ obtained by pushing up a_1 is homotopic to x_1^m . The loops a_1^- , a_2^+ and a_2^- are homotopic to $x_1^m x_2^{-1}$, $x_2^{-n} x_1$ and x_2^{-n} respectively. Thus we have the following Lin presentation of $\pi_1(E_{J(2m, 2n)})$:

$$\pi_1(E_{J(2m, 2n)}) = \langle x_1, x_2, \mu \mid \mu x_1^m \mu^{-1} = x_1^m x_2^{-1}, \mu x_2^{-n} x_1 \mu^{-1} = x_2^{-n} \rangle.$$

Definition 2.9. Let ρ be an $\mathrm{SL}_2(\mathbb{C})$ -representation of $\pi_1(E_K)$. We say that ρ is *metabelian* if ρ sends the commutator subgroup $[\pi_1(E_K), \pi_1(E_K)]$ to an abelian subgroup in $\mathrm{SL}_2(\mathbb{C})$. An $\mathrm{SL}_2(\mathbb{C})$ -representation ρ is called *irreducible* if the action of $\rho(\pi_1(E_K))$ on \mathbb{C}^2 does not have any invariant subspaces except for $\{0\}$ and \mathbb{C}^2 .

Proposition 2.10 (Proposition 1.1 in [Nag07] and Proposition 2.8 in [Yam13]). *Let ρ be an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of a knot group $\pi_1(E_K)$. For any Lin presentation $\pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle$, there exists a matrix $C \in \mathrm{SL}_2(\mathbb{C})$ such that*

$$C\rho(x_i)C^{-1} = \begin{bmatrix} z_i & 0 \\ 0 & z_i^{-1} \end{bmatrix} \quad (\forall i = 1, \dots, 2g) \quad \text{and} \quad C\rho(\mu)C^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Here each z_i is a root of unity whose order is a divisor of $\Delta_K(-1)$.

Remark 2.11. It is known that the number of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representations of $\pi_1(E_K)$ is equal to $(|\Delta_K(-1)| - 1)/2$.

2.3. Relation to the Reidemeister torsion. We refer to [KL99, Kit96] for the details on the relation between the twisted Alexander polynomial and the Reidemeister torsion. Here we review the Reidemeister torsion from Fox differential and restrict our attention to the Reidemeister torsion for E_K and $\mathrm{SL}_n(\mathbb{C})$ -representations $\rho_n = \sigma_n \circ \rho$ of $\pi_1(E_K)$.

Choose a presentation $\pi_1(E_K) = \langle g_1, \dots, g_k, r_1, \dots, r_{k-1} \rangle$ of deficiency 1. Let $\tilde{\rho}_n$ be the linear extension of ρ_n to the group ring $\mathbb{Z}[\pi_1(E_K)]$. By a similar argument to [Kit96, Proposition 3.1], one can show the following relation between the twisted Alexander polynomial and the Reidemeister torsion of a knot exterior.

Proposition 2.12. *Suppose that $1 \leq \ell \leq k$ such that*

$$\det(\tilde{\rho}_n(g_\ell - 1)) \neq 0 \quad \text{and} \quad \det\left(\tilde{\rho}_n\left(\frac{\partial r_i}{\partial g_j}\right)\right)_{\substack{1 \leq i \leq k-1 \\ 1 \leq j \leq k, j \neq \ell}} \neq 0$$

where $\partial r_i / \partial g_j$ is the Fox differential of r_i by g_j . Then the Reidemeister torsion of $\mathbb{T}_{K,\rho}$ is expressed, up to sign, as

$$(2) \quad \mathbb{T}_{K,\rho_n} = \frac{\det \left(\widetilde{\rho}_n \left(\frac{\partial r_i}{\partial g_j} \right) \right)_{\substack{1 \leq i \leq k-1 \\ 1 \leq j \leq k, j \neq \ell}}}{\det \widetilde{\rho}(g_\ell - 1)}.$$

Remark 2.13. We will use (2) instead of the definition of \mathbb{T}_{K,ρ_n} . We touch the well-definedness of \mathbb{T}_{K,ρ_n} without proofs.

- The right hand side of (2) is equal to $\Delta_{K,\rho_n}(1)$. Hence it is independent of the choice of ℓ . In the case of $n = 2N$, the right hand side of (2) is determined within sign.
- The assumption in Proposition 2.12 means that the twisted chain complex used to define the Reidemeister torsion is acyclic. For details on the definition of \mathbb{T}_{K,ρ_n} , see the review in [TY].

3. ASYMPTOTIC BEHAVIOR FOR METABELIAN REPRESENTATIONS

Here and subsequently, ρ denotes an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of a knot group $\pi_1(E_K)$. Choose a Lin presentation of $\pi_1(E_K)$:

$$\pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle.$$

By Proposition 2.10, up to conjugation, we may assume that ρ has the following form

$$\rho(x_i) = \begin{bmatrix} z_i & 0 \\ 0 & z_i^{-1} \end{bmatrix}, \quad \rho(\mu) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

where each z_i is a root of unity whose order is a divisor of $\Delta_K(-1)$.

Proposition 3.1. *The induced $\mathrm{SL}_{2N}(\mathbb{C})$ -representation ρ_{2N} is conjugate to the direct sum of metabelian representations as*

$$\rho_{2N} \underset{\text{conj.}}{\sim} \bigoplus_{j=1}^N \psi_j$$

where every ψ_j is a metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of $\pi_1(E_K)$, given by

$$(3) \quad \psi_j(x_i) = \begin{bmatrix} z_i^{1-2j} & 0 \\ 0 & z_i^{2j-1} \end{bmatrix}, \quad \psi_j(\mu) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Proof. Set $W_j = \text{span}_{\mathbb{C}} \langle x^{N+j-1} y^{N-j}, x^{N-j} y^{N+j-1} \rangle$ in V_{2N} as the 2-dimensional subspace spanned by $x^{N+j-1} y^{N-j}$ and $x^{N-j} y^{N+j-1}$. It follows from a direct computation that the restriction of ρ_{2N} to W_j is expressed as

$$\rho_{2N}|_{W_j}(x_i) = \begin{bmatrix} z_i^{1-2j} & 0 \\ 0 & z_i^{2j-1} \end{bmatrix}, \quad \rho_{2N}|_{W_j}(\mu) = (-1)^{N-j} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Taking conjugation by $\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$ if necessary, the restriction $\rho_{2N}|_{W_j}$ is conjugate to ψ_j given by (3). \square

By Proposition 3.1, one can see that ρ_{2N} has a periodicity in the direct sum decomposition. The asymptotic behavior of $\Delta_{K,\rho_{2N}}(t)$ is derived from this periodicity.

Theorem 3.2. *Let ρ be an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of $\pi_1(E_K)$. Then there exists metabelian representations ψ_1, \dots, ψ_p such that $\psi_1 = \rho$ and the limit of the leading coefficient of $\log \Delta_{K, \rho_{2N}}(t)$ is expressed as*

$$(4) \quad \lim_{N \rightarrow \infty} \frac{\log \Delta_{K, \rho_{2N}}(t)}{2N} = \frac{1}{2p} \sum_{j=1}^p \log \Delta_{K, \psi_j}(t).$$

Proof. The twisted Alexander polynomial has the invariance under conjugation of representations. We can consider the twisted Alexander polynomials $\Delta_{K, \rho_{2N}}(t)$ for $\mathrm{SL}_{2N}(\mathbb{C})$ -representations ρ_{2N} up to conjugation. Proposition 3.1 shows that ρ_{2N} turns into the direct sum of metabelian representations ψ_j . Under this decomposition of ρ_{2N} , it holds that

$$\Delta_{K, \rho_{2N}}(t) = \prod_{j=1}^N \Delta_{K, \psi_j}(t).$$

Let p be the l.c.m of orders of the roots of unity z_1, \dots, z_{2g} . Note that the l.c.m p is odd since the order of z_i is a divisor of the odd integer $\Delta_K(-1)$. It is easy seen that $\psi_{j+p} = \psi_j$ for all j . This implies that $\{\Delta_{K, \psi_j}(t)\}_{j=1,2,\dots}$ is a periodic sequence of rational functions in t with period p . Hence it follows from [Yam, Lemma 3.11] that

$$\lim_{N \rightarrow \infty} \frac{\log \Delta_{K, \rho_{2N}}(t)}{2N} = \lim_{N \rightarrow \infty} \frac{\sum_{j=1}^N \log \Delta_{K, \psi_j}(t)}{2N} = \frac{1}{2p} \sum_{j=1}^p \log \Delta_{K, \psi_j}(t).$$

□

Remark 3.3. The period p is the l.c.m. of divisors of $\Delta_K(-1)$. Hence p is also a divisor of $\Delta_K(-1)$.

Assume that $\Delta_{K, \psi_j}(1) \neq 0$ for all $j = 1, \dots, p$. Under this assumption, we can also consider the asymptotic behavior of the Reidemeister torsion from that of the twisted Alexander polynomial.

Corollary 3.4. *Suppose that $\Delta_{K, \psi_j}(1) \neq 0$ for $1 \leq j \leq p$. Then we can define the Reidemeister torsion $\mathbb{T}_{K, \rho_{2N}}$ for all N . The growth order of $\log |\mathbb{T}_{K, \rho_{2N}}|$ is the same as $2N$. The limit of the leading coefficient of $\log |\mathbb{T}_{K, \rho_{2N}}|$ is expressed as*

$$(5) \quad \lim_{N \rightarrow \infty} \frac{\log |\mathbb{T}_{K, \rho_{2N}}|}{2N} = \frac{1}{2p} \sum_{j=1}^p \log |\Delta_{K, \psi_j}(1)|.$$

Proof. Proposition 3.1 shows that ρ_{2N} is conjugate to the direct sum $\oplus_{j=1}^N \psi_j$ of ψ_j given by (3). The twisted Alexander polynomial $\Delta_{K, \rho_{2N}}(t)$ turns into the product $\prod_{j=1}^N \Delta_{K, \psi_j}(t)$. Since $\psi_{j+p} = \psi_j$ for any j , the assumption $\Delta_{K, \psi_j}(1) \neq 0$ implies that $\Delta_{K, \rho_{2N}}(1) \neq 0$ for all N . The eigenvalues of $\rho(\mu)$ are $\pm \sqrt{-1}$. One can see that $\det(\rho_{2N}(\mu) - \mathbf{1}) = 2^N$. By definition, it holds for all N that

$$\det \left(\widetilde{\rho_{2N}} \left(\frac{\partial r_k}{\partial x_l} \right) \right)_{\substack{1 \leq k \leq 2g \\ 1 \leq l \leq 2g}} \neq 0$$

where $r_k = \mu a_k^+ \mu^{-1} (a_k^-)^{-1}$ in a Lin presentation.

It follows from Proposition 2.12 that $\mathbb{T}_{K, \rho_{2N}} = \Delta_{K, \rho_{2N}}(1)$ for any N . Taking $t = 1$, our claim (5) follows from the equality (4). □

4. GENUS ONE TWO-BRIDGE KNOTS

Genus one two-bridge knots are double twist knots $J(2m, \pm 2n)$ in Hoste-Shanahan's notation [HS04], see also [Lin01]. Here m and n are positive integers. The Alexander polynomial of $K = J(2m, \pm 2n)$ is

$$\Delta_K(t) = mnt^2 + (1 \mp 2mn)t + mn.$$

4.1. The case of $K = J(2m, 2n)$. Then $(|\Delta_K(-1)| - 1)/2 = 2mn - 1$ and hence there are $2mn - 1$ irreducible metabelian representations $\rho_i: \pi_1(E_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$, where $1 \leq i \leq 2mn - 1$.

By Example 2.8 a Lin presentation of $K = J(2m, 2n)$ is

$$\pi_1(E_K) = \langle x_1, x_2, \mu \mid \mu x_1^m \mu^{-1} = x_1^m x_2^{-1}, \mu x_2^{-n} x_1 \mu^{-1} = x_2^{-n} \rangle.$$

Up to conjugation, we may assume that

$$\rho_i(x_1) = \begin{bmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{bmatrix}, \quad \rho_i(x_2) = \begin{bmatrix} \xi^{2mi} & 0 \\ 0 & \xi^{-2mi} \end{bmatrix}, \quad \rho_i(\mu) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $\xi = e^{2\pi\sqrt{-1}/(4mn-1)}$ and $1 \leq i \leq 2mn - 1$. By Proposition 3.1, we have

$$(\rho_i)_{2N} = \sigma_{2N} \circ \rho_i \underset{\text{conj.}}{\sim} \bigoplus_{j=1}^N \psi_{i,j}$$

where

$$\psi_{i,j}(x_1) = \begin{bmatrix} \xi^{i(1-2j)} & 0 \\ 0 & \xi^{i(2j-1)} \end{bmatrix}, \quad \psi_{i,j}(x_2) = \begin{bmatrix} \xi^{2mi(1-2j)} & 0 \\ 0 & \xi^{2mi(2j-1)} \end{bmatrix}, \quad \psi_{i,j}(\mu) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $r_1 = \mu x_1^m \mu^{-1} x_2 x_1^{-m}$ and $r_2 = \mu x_2^{-n} x_1 \mu^{-1} x_2^n$. We have

$$\Delta_{K, \psi_{i,j}}(t) = \det \begin{bmatrix} \Phi_{\psi_{i,j}}(\frac{\partial r_1}{\partial x_1}) & \Phi_{\psi_{i,j}}(\frac{\partial r_1}{\partial x_2}) \\ \Phi_{\psi_{i,j}}(\frac{\partial r_2}{\partial x_1}) & \Phi_{\psi_{i,j}}(\frac{\partial r_2}{\partial x_2}) \end{bmatrix} / \det \Phi_{\psi_{i,j}}(1 - \mu),$$

where

$$\begin{aligned} \frac{\partial r_1}{\partial x_1} &= \mu(1 - x_1^m \mu^{-1} x_2 x_1^{-m})(1 + x_1 + \cdots + x_1^{m-1}) \\ &= \mu(1 - \mu^{-1})(1 + x_1 + \cdots + x_1^{m-1}), \\ \frac{\partial r_1}{\partial x_2} &= \mu x_1^m \mu^{-1}, \\ \frac{\partial r_2}{\partial x_1} &= \mu x_2^{-n}, \\ \frac{\partial r_2}{\partial x_2} &= \mu(-1 + x_2^{-n} x_1 \mu^{-1} x_2^n)(1 + x_2^{-1} + \cdots + x_2^{-(n-1)})x_2^{-1} \\ &= \mu(-1 + \mu^{-1})(1 + x_2^{-1} + \cdots + x_2^{-(n-1)})x_2^{-1}. \end{aligned}$$

For $k \geq 0$ and $b \in \mathbb{C}$ let $\delta_k(b) = 1 + b + \cdots + b^k$. Note that $(1 + b^{k+1})\delta_k(b) = \delta_{2k+1}(b)$.

Proposition 4.1. *The twisted Alexander polynomial $\Delta_{K, \psi_{i,j}}(t)$ is expressed as follows.*

(a) *If $\xi^{i(2j-1)} \neq 1$, then*

$$\Delta_{K, \psi_{i,j}}(t) = (\xi^{i(2j-1)})^{1+m-2mn} \left(\delta_{m-1}(\xi^{i(2j-1)}) \delta_{n-1}(\xi^{2mi(2j-1)}) \right)^2 (t^2 + 1).$$

(b) If $\xi^{i(2j-1)} = 1$, then

$$\Delta_{K,\psi_{i,j}}(t) = \frac{m^2 n^2 t^4 + (2m^2 n^2 - 4mn + 1)t^2 + m^2 n^2}{t^2 + 1}.$$

Proof. For brevity, we set $a = \xi^{i(1-2j)}$. Then $\psi_{i,j}$ is written as

$$\psi_{i,j}(x_1) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad \psi_{i,j}(x_2) = \begin{bmatrix} a^{2m} & 0 \\ 0 & a^{-2m} \end{bmatrix}, \quad \psi_{i,j}(\mu) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \Phi_{\psi_{i,j}}\left(\frac{\partial r_1}{\partial x_1}\right) &= \delta_{m-1}(a) \begin{bmatrix} -1 & ta^{1-m} \\ -t & -a^{1-m} \end{bmatrix}, \\ \Phi_{\psi_{i,j}}\left(\frac{\partial r_1}{\partial x_2}\right) &= \begin{bmatrix} a^{-m} & 0 \\ 0 & a^m \end{bmatrix}, \\ \Phi_{\psi_{i,j}}\left(\frac{\partial r_2}{\partial x_1}\right) &= \begin{bmatrix} 0 & ta^{2mn} \\ -ta^{-2mn} & 0 \end{bmatrix}, \\ \Phi_{\psi_{i,j}}\left(\frac{\partial r_2}{\partial x_2}\right) &= \delta_{n-1}(a^{2m}) \begin{bmatrix} a^{-2mn} & -ta^{2m} \\ ta^{-2mn} & a^{2m} \end{bmatrix}. \end{aligned}$$

Hence, by a direct calculation, we have

$$\begin{aligned} \det \left[\Phi_{\psi_{i,j}}\left(\frac{\partial r_k}{\partial x_l}\right) \right]_{1 \leq k, l \leq 2} &= t^2 - (a^m + a^{1-4mn})(a^{2mn} + 1)\delta_{m-1}(a)\delta_{n-1}(a^{2m})t^2 \\ &\quad + a^{1+m-2mn}(\delta_{m-1}(a)\delta_{n-1}(a^{2m}))^2(t^2 + 1)^2. \end{aligned}$$

Since $\xi^{4mn-1} = 1$, we have $a^{4mn-1} = 1$. This implies that

$$\begin{aligned} (a^m + a^{1-4mn})(a^{2mn} + 1)\delta_{m-1}(a)\delta_{n-1}(a^{2m}) &= (a^m + 1)\delta_{m-1}(a)(a^{2mn} + 1)\delta_{n-1}(a^{2m}) \\ &= \delta_{2m-1}(a)\delta_{2n-1}(a^{2m}) \\ &= \delta_{4mn-1}(a) \\ &= \begin{cases} \frac{a^{4mn}-1}{a-1} = 1 & \text{if } a \neq 1 \\ 4mn & \text{if } a = 1 \end{cases}. \end{aligned}$$

Hence, we obtain

$$\det \left[\Phi_{\psi_{i,j}}\left(\frac{\partial r_k}{\partial x_l}\right) \right]_{1 \leq k, l \leq 2} = \begin{cases} a^{1+m-2mn}(\delta_{m-1}(a)\delta_{n-1}(a^{2m}))^2(t^2 + 1)^2 & \text{if } a \neq 1 \\ m^2 n^2 (t^2 + 1)^2 - (4mn - 1)t^2 & \text{if } a = 1. \end{cases}$$

The lemma follows, since $\det \Phi_{\psi_{i,j}}(1 - \mu) = t^2 + 1$. \square

Remark 4.2. Note that $a = 1$ corresponds to an abelian representation and $m^2 n^2 (t^2 + 1)^2 - (4mn - 1)t^2 = \Delta_{J(2m, 2n)}(\sqrt{-1}t)\Delta_{J(2m, 2n)}(-\sqrt{-1}t)$.

Lemma 4.3. Suppose λ is a root of unity whose order is an odd integer q . Let k be an positive integer coprime with q . Then

$$\prod_{\substack{1 \leq j \leq q, \\ j \neq (q+1)/2}} \delta_{k-1}(\lambda^{2j-1}) = 1.$$

Proof. Consider $1 \leq j \leq q$. Note that $\lambda^{2j-1} = 1$ if and only if $j = \frac{q+1}{2}$. We have

$$\prod_{\substack{1 \leq j \leq q, \\ j \neq (q+1)/2}} \delta_{k-1}(\lambda^{2j-1}) = \prod_{\substack{1 \leq j \leq q, \\ j \neq (q+1)/2}} \frac{\lambda^{k(2j-1)} - 1}{\lambda^{2j-1} - 1}.$$

Since $q = \text{order}(\lambda)$ is an odd integer, for any integer l co-prime with q we have that the map $j \mapsto l(2j-1) \pmod{q}$ gives an bijection from the set $\{1 \leq j \leq q \mid j \neq (q+1)/2\}$ to $\{1, 2, \dots, q-1\}$. In particular, we have

$$\prod_{\substack{1 \leq j \leq q, \\ j \neq (q+1)/2}} (\lambda^{l(2j-1)} - 1) = (\lambda - 1)(\lambda^2 - 1) \cdots (\lambda^{q-1} - 1).$$

Since k is coprime with q , the denominator coincides with the numerator in the product of $\delta_{k-1}(\lambda^{2j-1}) = \frac{\lambda^{k(2j-1)} - 1}{\lambda^{2j-1} - 1}$. This is our claim. \square

Let $q_i = \frac{4mn-1}{\gcd(4mn-1, i)}$. Note that ξ^i is a root of unity whose order equals to q_i . Moreover, for $1 \leq j \leq q_i$ we have $\xi^{i(2j-1)} = 1$ if and only if $j = \frac{q_i+1}{2}$.

Since the orders of ξ^i and ξ^{2mi} are equal to q_i which is coprime with both m and n , Lemma 4.3 implies that

$$\prod_{\substack{1 \leq j \leq q_i, \\ j \neq (q_i+1)/2}} \delta_{m-1}(\xi^{i(2j-1)}) = 1 \quad \text{and} \quad \prod_{\substack{1 \leq j \leq q_i, \\ j \neq (q_i+1)/2}} \delta_{n-1}(\xi^{2mi(2j-1)}) = 1.$$

Note that

$$\prod_{\substack{1 \leq j \leq q_i, \\ j \neq (q_i+1)/2}} \xi^{i(2j-1)} = \prod_{1 \leq j \leq q_i} \xi^{i(2j-1)} = 1.$$

We obtain the following theorem from the above computations.

Theorem 4.4. Set $q_i = \frac{4mn-1}{\gcd(4mn-1, i)}$. The limit of $(\log \Delta_{J(2m, 2n), (\rho_i)_{2N}}(t)) / (2N)$ is expressed as

$$\lim_{N \rightarrow \infty} \frac{\log \Delta_{J(2m, 2n), (\rho_i)_{2N}}(t)}{2N} = \frac{1}{2q_i} \log \frac{m^2 n^2 t^4 + (2m^2 n^2 - 4mn + 1)t^2 + m^2 n^2}{(t^2 + 1)^2} + \frac{1}{2} \log(t^2 + 1).$$

Proof. Since $\Delta_{K, (\rho_i)_{2N}}(t) = \prod_{j=1}^N \Delta_{K, \psi_{i,j}}(t)$, Proposition 4.1 implies

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{\log \Delta_{J(2m, 2n), (\rho_i)_{2N}}(t)}{2N} \\ &= \frac{1}{2q_i} \sum_{j=1}^{q_i} \log \Delta_{J(2m, 2n), \psi_{i,j}}(t) \\ &= \frac{1}{2q_i} \left(\log \frac{m^2 n^2 t^4 + (2m^2 n^2 - 4mn + 1)t^2 + m^2 n^2}{t^2 + 1} + (q_i - 1) \log(t^2 + 1) \right) \\ &= \frac{1}{2q_i} \log \frac{m^2 n^2 t^4 + (2m^2 n^2 - 4mn + 1)t^2 + m^2 n^2}{(t^2 + 1)^2} + \frac{1}{2} \log(t^2 + 1). \end{aligned}$$

\square

Corollary 4.5. The leading coefficient of $\log |\mathbb{T}_{J(2m, 2n), (\rho_i)_{2N}}|$ converges as follows:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\log \mathbb{T}_{J(2m, 2n), (\rho_i)_{2N}}}{2N} &= \frac{1}{q_i} \log \frac{2mn - 1}{2} + \frac{1}{2} \log 2 \\ &= \frac{\gcd(4mn - 1, i)}{4mn - 1} \log \frac{2mn - 1}{2} + \frac{1}{2} \log 2. \end{aligned}$$

In particular, the growth order of $\log |\mathbb{T}_{J(2m,2n),(\rho_i)_{2N}}|$ is the same as $2N$.

Proof. Since $\Delta_{J(2m,2n),\psi_{i,j}}(1) \neq 0$ for $1 \leq \forall j \leq q_i$, we can apply Corollary 3.4 to this situation. \square

4.2. The case of $K = J(2m, -2n)$. Since $(|\Delta_K(-1)| - 1)/2 = 2mn$, there are $2mn$ irreducible metabelian representations $\rho_i: \pi_1(E_K) \rightarrow \mathrm{SL}_2(\mathbb{C})$, where $1 \leq i \leq 2mn$.

A Lin presentation of $K = J(2m, 2n)$ is

$$\pi_1(E_K) = \langle x_1, x_2, \mu \mid \mu x_1^m \mu^{-1} = x_1^m x_2^{-1}, \mu x_2^n x_1 \mu^{-1} = x_2^n \rangle.$$

Up to conjugation, we may assume that

$$\rho_i(x_1) = \begin{bmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{bmatrix}, \quad \rho_i(x_2) = \begin{bmatrix} \xi^{2mi} & 0 \\ 0 & \xi^{-2mi} \end{bmatrix}, \quad \rho_i(\mu) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $\xi = e^{2\pi \sqrt{-1}/(4mn+1)}$ and $1 \leq i \leq 2mn$. Then By Proposition 3.1 shows that

$$(\rho_i)_{2N} \sim_{\text{conj.}} \bigoplus_{j=1}^N \psi_{i,j},$$

$$\psi_{i,j}(x_1) = \begin{bmatrix} \xi^{i(1-2j)} & 0 \\ 0 & \xi^{i(2j-1)} \end{bmatrix}, \quad \psi_{i,j}(x_2) = \begin{bmatrix} \xi^{2mi(1-2j)} & 0 \\ 0 & \xi^{2mi(2j-1)} \end{bmatrix}, \quad \psi_{i,j}(\mu) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

By a similar argument to the case of $J(2m, 2n)$, we obtain the asymptotic behavior of the twisted Alexander polynomial for $J(2m, -2n)$.

Theorem 4.6. Let $q_i = \frac{4mn+1}{\gcd(4mn+1, i)}$. For $1 \leq i \leq 2mn$, it holds that

$$\lim_{N \rightarrow \infty} \frac{\log \Delta_{K, (\rho_i)_{2N}}(t)}{2N} = \frac{1}{2q_i} \log \frac{m^2 n^2 t^4 + (2m^2 n^2 + 4mn + 1)t^2 + m^2 n^2}{(t^2 + 1)^2} + \frac{1}{2} \log(t^2 + 1).$$

Proof. The sequence of $\Delta_{K, (\rho_i)_{2N}}(t)$ has the period q_i . The limit of $\frac{\log \Delta_{K, (\rho_i)_{2N}}(t)}{2N}$ turns into

$$\begin{aligned} & \frac{1}{2q_i} \sum_{j=1}^{q_i} \log \Delta_{K, \psi_{i,j}}(t) \\ &= \frac{1}{2q_i} \left(\log \frac{m^2 n^2 t^4 + (2m^2 n^2 + 4mn + 1)t^2 + m^2 n^2}{t^2 + 1} + (q_i - 1) \log(t^2 + 1) \right) \\ &= \frac{1}{2q_i} \log \frac{m^2 n^2 t^4 + (2m^2 n^2 + 4mn + 1)t^2 + m^2 n^2}{(t^2 + 1)^2} + \frac{1}{2} \log(t^2 + 1). \end{aligned}$$

\square

We also have the similar corollary to the case of $J(2m, 2n)$.

Corollary 4.7. The leading coefficient of $\log |\mathbb{T}_{K, (\rho_i)_{2N}}|$ converges as follows:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\log \mathbb{T}_{K, (\rho_i)_{2N}}}{2N} &= \frac{1}{q_i} \log \frac{2mn + 1}{2} + \frac{1}{2} \log 2 \\ &= \frac{\gcd(4mn + 1, i)}{4mn + 1} \log \frac{2mn + 1}{2} + \frac{1}{2} \log 2. \end{aligned}$$

In particular, the growth order of $\log |\mathbb{T}_{K, (\rho_i)_{2N}}|$ is the same as $2N$.

Remark 4.8. Since the twist knot K_n in [TY] is the mirror image of $J(2, -2n)$, we have

$$\lim_{N \rightarrow \infty} \frac{\log \mathbb{T}_{K_n, (\rho_i)_{2N}}}{2N} = \frac{\gcd(4n+1, i)}{4n+1} \log \frac{2n+1}{2} + \frac{1}{2} \log 2.$$

Together with [Yam, Theorem 3.7] and $\text{order}(\rho_i(\mu)) = 4$, one can also obtain the asymptotic behavior of the higher dimensional Reidemeister torsion for the graph manifold obtained by 4-surgery along K_n as in [TY]. Compared with [TY, Theorem 4.4], it can be seen that the coefficient $\gcd(4n+1, i)/(4n+1)$ is determined by the torus knot exterior in the graph manifold obtained by 4-surgery along K_n .

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